

$y_1 = e^{-x}$ is a solution of $xy'' + (2x+3)y' + (x+3)y = 0$.

SCORE: ____ / 8 PTS

Find a second linearly independent solution.

$$y_2 = ve^{-x}$$

$$y_2' = \underline{v'e^{-x} - ve^{-x}}$$

$$y_2'' = \underline{v''e^{-x} - 2v'e^{-x} + ve^{-x}}$$

① POINT EACH
UNLESS OTHERWISE NOTED

$$xy_2'' + (2x+3)y_2' + (x+3)y_2$$

$$= \underline{xe^{-x}v'' - 2xe^{-x}v'} + \underline{xe^{-x}v} \left(\frac{1}{2}\right)$$

$$+ \underline{(2x+3)e^{-x}v'} - (2x+3)e^{-x}v \left(\frac{1}{2}\right)$$

$$+ \underline{(x+3)e^{-x}v} \left(\frac{1}{2}\right)$$

$$= \underline{xe^{-x}v'' + 3e^{-x}v'} = 0$$

$$xv'' + 3v' = 0$$

$$u = v' \quad \left(\frac{1}{2}\right) \underline{xu' + 3u} = 0$$

$$x \frac{du}{dx} + 3u = 0$$

$$\left(\frac{1}{2}\right) \underline{\int \frac{1}{u} du = \int -\frac{3}{x} dx}$$

$$\ln|u| = -3 \ln|x|$$

$$\underline{u = x^{-3}}$$

$$v' = x^{-3}$$

$$\left(\frac{1}{2}\right) \underline{v = -\frac{1}{2}x^{-2}}$$

$$\underline{y_2 = x^{-2}e^{-x}}$$

Find the general solution of the homogeneous linear differential equation $2y''' - 3y'' + 18y' + 10y = 0$.

SCORE: ___ / 6 PTS

$$2r^3 - 3r^2 + 18r + 10 = 0$$

$$r = \pm \frac{1, 2, 5, 10}{1, 2} = \pm 1, 2, 5, 10, \frac{1}{2}, \frac{5}{2}$$

2 SIGN CHANGES \rightarrow 2 or 0 POSITIVE ROOTS

$$-2r^3 - 3r^2 - 18r + 10 = 0$$

1 SIGN CHANGE \rightarrow 1 NEGATIVE ROOT

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 18 & 10 \\ & & -2 & 5 & -23 \\ \hline & 2 & -5 & 23 & -13 \end{array}$$

ALTERNATING,
NO ROOTS < -1

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -3 & 18 & 10 \\ & & -1 & 2 & -10 \\ \hline & 2 & -4 & 20 & 0 \end{array} \checkmark$$

$$(r + \frac{1}{2})(2r^2 - 4r + 20) = 0$$

$$2(r + \frac{1}{2})(r^2 - 2r + 10) = 0$$

$$r = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm 6i}{2}$$

$$= 1 \pm 3i$$

① POINT EACH

$$y = \underbrace{c_1 e^{-\frac{1}{2}x}}_{\text{}} + \underbrace{c_2 e^x \cos 3x}_{\text{}} + \underbrace{c_3 e^x \sin 3x}_{\text{}}$$

Find the general solutions of the following homogeneous linear differential equations.

SCORE: ____ / 5 PTS

[a] $16y'' - 24y' + 9y = 0$

$$\underline{16r^2 - 24r + 9 = 0} \quad \left(\frac{1}{2}\right)$$

$$(4r - 3)^2 = 0$$

$$\underline{r = \frac{3}{4}, \frac{3}{4}} \quad \left(\frac{1}{2}\right)$$

$$y = \underline{Ae^{\frac{3}{4}x} + Bxe^{\frac{3}{4}x}} \quad (1)$$

[b] $4x^2y'' + 12xy' + 5y = 0$

$$\underline{4r^2 + 8r + 5 = 0} \quad (1)$$

$$r = \frac{-8 \pm \sqrt{64 - 80}}{8} = \frac{-8 \pm 4i}{8}$$

$$= \underline{-1 \pm \frac{1}{2}i} \quad (1)$$

$$y = \underline{Ax^{-1} \cos\left(\frac{1}{2} \ln x\right) + Bx^{-1} \sin\left(\frac{1}{2} \ln x\right)} \quad (1)$$

Consider the non-homogeneous linear differential equation $2x^2y'' + 7xy' - 3y = \frac{4}{x}$.

SCORE: ____ / 11 PTS

[a] If $y = \frac{2}{x}$ is a particular solution of the equation, find the value of A .

$$\underline{2x^2(4x^{-3}) + 7x(-2x^{-2}) - 3(2x^{-1})} = 8x^{-1} - 14x^{-1} - 6x^{-1}$$
$$= -12x^{-1} \quad \underline{A = -12}$$

[b] Using linearity, find a particular solution of $2x^2y'' + 7xy' - 3y = \frac{4}{x}$.

$$\frac{4}{x} = -\frac{1}{3}\left(\frac{12}{x}\right)$$
$$y_p = -\frac{1}{3}\left(\frac{2}{x}\right) = \underline{-\frac{2}{3}x^{-1}}$$

① POINT EACH
UNLESS OTHERWISE
NOTED

[c] Using superposition, find the general solution of $2x^2y'' + 7xy' - 3y = \frac{4}{x}$.

$$\underline{2r^2 + 5r - 3 = 0}$$
$$(2r - 1)(r + 3) = 0$$
$$\underline{r = \frac{1}{2}, -3}$$

$$y = \underline{-\frac{2}{3}x^{-1} + Ax^{\frac{1}{2}} + Bx^{-3}} \quad \textcircled{2}$$

[d] Solve the initial value problem $2x^2y'' + 7xy' - 3y = \frac{4}{x}$, $y(1) = \frac{4}{3}$, $y'(1) = -\frac{1}{3}$.

$$y(1) = -\frac{2}{3} + A + B = \frac{4}{3} \rightarrow \underline{A + B = 2}$$

$$y' = \underline{\frac{2}{3}x^{-2} + \frac{1}{2}Ax^{-\frac{1}{2}} - 3Bx^{-4}}$$

$$y'(1) = \frac{2}{3} + \frac{1}{2}A - 3B = -\frac{1}{3} \rightarrow \underline{\frac{1}{2}A - 3B = -1}$$

$$A - 6B = -2$$

$$A + B = 2$$

$$\rightarrow -7B = -4$$

$$B = \frac{4}{7}$$

$$A = \frac{10}{7}$$

$$y = \underline{-\frac{2}{3}x^{-1} + \frac{10}{7}x^{\frac{1}{2}} + \frac{4}{7}x^{-3}}$$